SS 2016 Dr. Milena Pabiniak
Marc Kegel

Differential Geometry

Homework 5

Mandatory Exercise 1. (10 points)
Consider the following differential forms on $\mathbb{R}^3$.

$$\omega_1 := (x^2 - yz)\,dx + (y^2 - xz)\,dy - xy\,dz$$
$$\omega_2 := \omega_1 + 2xy\,dz$$
$$\omega_3 := 2xz\,dy \wedge dz + dz \wedge dx - (z^2 + e^x)\,dx \wedge dy$$

A differential form ω is called **closed** if $d\omega = 0$ and **exact** if there exists a differential form η with $d\eta = \omega$. Which of these forms are closed, which are exact?

Mandatory Exercise 2. (10 points)
Let ω, ω_1 and ω_2 be k-forms on a smooth manifold M. Show that:

(a) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

(b) If $f\colon N \to M$ is a smooth map, then $d(f^*\omega) = f^*d\omega$.

Suggested Exercise 1. (0 points)
Given a k-form ω on a smooth manifold M. We can define its exterior derivative $d\omega$ without using local coordinates: given $k+1$ vector fields $X_1, \dots, X_{k+1}$ on M, define

$$d\omega\big(X_1, \dots, X_{k+1}\big) := \sum_{i=1}^{k+1} (-1)^{i-1} X_i \cdot \omega\big(X_1, \dots, \hat{X}_i, \dots, X_{k+1}\big) + \sum_{i<j} (-1)^{i+j} \omega\big([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}\big).$$

(a) Show that $d\omega$ is in fact a $k+1$ form.

(b) Show that the above definition of $d\omega$ coincides with the definition from the lecture.

Suggested Exercise 2. (0 points)

(a) Consider the 1-form $\alpha := f^1\,dx + f^2\,dy + f^3\,dz$ on $\mathbb{R}^3$. Show that

$$d\alpha = g^1\,dy \wedge dz + g^2\,dz \wedge dx + g^3\,dx \wedge dy,$$

where $(g^1, g^2, g^3) = \mathrm{curl}(f^1, f^2, f^3)$.

(b) Consider the 2-form $\omega = f^1\,dy \wedge dz + f^2\,dz \wedge dx + f^3\,dx \wedge dy$, on $\mathbb{R}^3$. Show that

$$d\omega = \mathrm{div}(f^1, f^2, f^3)\,dx \wedge dy \wedge dz.$$

Suggested Exercise 3. (0 points)
Let $\omega \in \Omega^1(S^2)$ be a differential 1-form such that for any $\phi \in SO(3)$ it holds that $\phi^*\omega = \omega$. Show that $\omega = 0$. Hint: Take a point $p \in S^2$ and look only at those $\phi \in SO(3)$ which fix p, and at the equation $(\phi^*\omega)_p = \omega_p$. How does $d\phi_p$ act on the tangent space T_pS^2?

Suggested Exercise 4. (0 points)
Let V be a vector space. The unique possibe contraction on $V \otimes V^*$ is $c_{1,1} \colon V \otimes V^* \to \mathbb{R}$. Show that $c_{1,1}$ is the trace when one views $V \otimes V^*$ as $Lin(V, V)$.

Suggested Exercise 5. (0 points)
Let $f \colon M \to N$ be a smooth map and α and β be forms on N.

(a) $f^*(\alpha + \beta) = f^*\alpha + f^*\beta$.

(b) $f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta)$. Note that viewing smooth functions as 0-forms the above formula gives $f^*(g\alpha) = (g \circ f) f^*\alpha = (f^*g)(f^*\alpha)$ for any smooth function $g \colon N \to \mathbb{R}$.

(c) $g^*(f^*\alpha) = (f \circ g)^*\alpha$ for any smooth map $g \colon P \to M$.

Hand in: Monday 23th May
in the exercise session
in Seminar room 2, MI